Ordering relations, partitions and Aristotelian diagrams

In mathematics, the general notion of scalarity is most naturally associated with *ordering relations*, such as partial orders and total orders. In this paper I will investigate such ordering relations in terms of the type of partition they induce of their underlying logical space, and in terms of the Aristotelian diagrams that one can construct for them. The overall argumentation will proceed in five main steps.

First of all, I will briefly review the most well-known Aristotelian diagram, viz. the square of opposition for the *categorical statements* of syllogistics. These statements do not seem to be directly related to an underlying ordering relation. They induce a partition of logical space into three cells: (i) 'all S are P', (ii) 'some but not all S are P' and (iii) 'no S are P'. (Consequently, the Boolean closure of this square of opposition contains $2^3 - 2 = 6$ contingent statements, i.e. it is a hexagon.) The three cells of this partition constitute a *total ordering* of logical space.

Secondly, we turn to two Aristotelian diagrams for *propositional logic*, viz. the classical square of opposition for the complex propositions p & q, p v q and their negations, and the 'degenerate' square for the atomic propositions p, q and their negations. Again, these propositions are not directly related to an underlying ordering relation. The degenerate square induces a partition of logical space into four cells: (i) p & q, (ii) p & ~q, (iii) ~p & q and (iv) ~p & ~q. (Consequently, the Boolean closure of this square contains $2^4 - 2 = 14$ contingent statements, i.e. it is a rhombic dodecahedron.) The four cells of this partition do not constitute any ordering of logical space, but rather exhibit a high degree of *symmetry* (they arise out of the interaction between two independent bipartitions: p/~p and q/~q.)

Thirdly, we consider the hexagon of opposition for a *total ordering* relation, which was already studied by R. Blanché in the 1950s. This hexagon induces a partition with three cells: (i) $x < y$, (ii) $x = y$ and (iii) $x > y$. (This shows that the Boolean closure of this hexagon is the hexagon itself, i.e. this hexagon is already Boolean closed.) The cells of this partition clearly reflect the *total ordering* that induced them.

Fourthly, we move from total ordering relations to *partial ordering* relations, by dropping the axiom of completeness (which requires that $x < y$ or $x > y$ for all distinct x, y). Hence, in comparison to the tripartition induced by a total ordering, we obtain a partition with one additional cell: (iv) x and y are incomparable. (Consequently, the Boolean closure of the hexagon for a partial ordering is a rhombic dodecahedron, with $2^4 - 2 = 14$ contingent statements.) The cells of this quadripartition clearly reflect the *partial ordering* that induced them.

Fifthly, I will shift back to *total ordering* relations, and examine the axiom of *transitivity* more closely. This axiom crucially involves three elements: if $x < y$ and $y < z$, then $x < z$. We can thus construct diagrams that simultaneously contain statements of the form x ? y, y ? z and x ? z, where ? is one of {<, =, >}. I will show that (and explain how) these statements induce a partition of logical space into exactly 13 cells. (The Boolean closure of such a diagram would thus contain $2^{13} - 2 = 8.190$ contingent statements, which is far too many to be actually drawn.) The 13 cells of this partition display a high degree of *symmetry*: there are 3! = 6 cells of the form $x < y < z$ (involving no identities), there are 6 additional cells of the form $x < y = z$ (involving one identity), and finally, there is one cell of the form $x = y = z$ (involving three identities). To further emphasize the symmetry involved in this 13-partition, I will also draw a connection with the notion of a *permutahedron* from geometric combinatorics.

To conclude, this paper has explored the intricate relationship between the order manifested in an Aristotelian diagram and the order manifested in the partition induced by that diagram. This relationship can be summarized by means of the following table:

Aristotelian diagram	induced partition	
not order-based	order-based	(cf. step 1)
not order-based	not order-based	(cf. step 2)
order-based	order-based	(cf. steps 3, 4)
order-based	not order-based	(cf. step 5)