Logic of colours Logica der kleuren

The mereological algebra of colours

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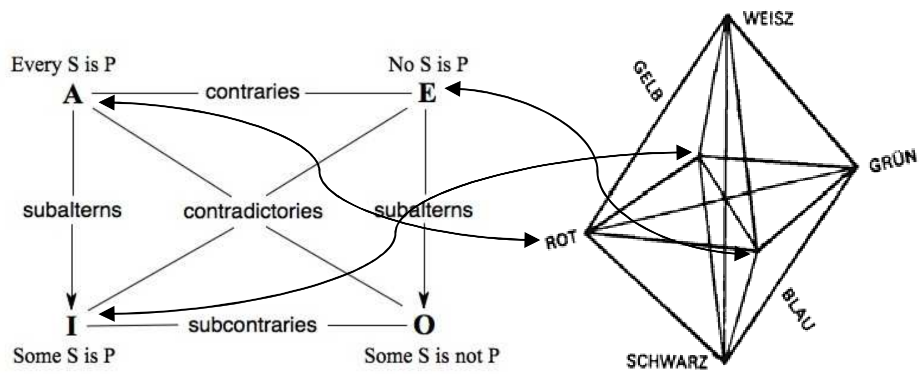
0. Introduction

(1) structure:

(1) definitions of oppositions in predicate logic, using a Smessaert-type bitstring-analysis with a string consisting of three values per quantifier.

(2) a mereological algebra of colours as an idealized binary basis for colour cognition.

(3) conclusion: isomorphism of the predicate logic and colour algebras



Square of Oppositions (Boethius, 5th century) Höfler (1897)

(4) Natural language application: parallels between the two linguistic domains of application of the respective algebras:

- Evolution sequence of terms for quantifier and colour oppositions
- natural versus non-natural quantifiers (*nand/*nall) and colour terms (yellow vs. "cyane"): cognitive complexity and the generalized O-corner problem.

1. Predicate Logic (based on Smessaert 2009)

(1)

U = {a,b,c,d,e,f,g,h}
D = {a,b,c,d} [[book]]
P1 = {e,f} [[be asleep]]
P2 = {c,d,g,h} [[be in English]]
P3 = {a,b,c,d,g,h} [[be worth reading]]
(Smessaert 2009: 304)

(2)

[ ]D:

[x1 . . . xn]D ≡ {X ⊆ U : X ∩D = {x1, . . . , xn}}

[a]D ≡ {X ⊆ U : X ∩D = {a}} [[the book a]]

[abc]D ≡ {X ⊆ U : X ∩D = {a,b,c}} [[the books a, b and c]]

[ ]D ≡ {X ⊆ U : X ∩D = ∅} [[neither a nor b nor c nor d]]

P1 ∈ [ ]D [[neither book a nor b nor c nor d is asleep]]

P2 ∈ [cd]D [[the books c and d are in English]]

P3 ∈ [abcd]D [[the books a, b, c and d are worth reading]]"

(3)

D*n* ≡ {X ⊆ U :/ X ∩ D /= n}

D0 ≡ [ ]*D*

D3 ≡ [abc]*D* ∪ [abd]*D* ∪ [acd]*D* ∪ [bcd]*D*

(4) Scalar structure (partition of the Powerset of the Universe)

D0	D1	D2	D3	D4
[ ]	[a]	[ab]	[abc]	[abcd]
	[b]	[ac]	[abd]	
	[c]	[ad]	[acd]	
	[d]	[bc]	[bcd]	
		[bd]		
		[cd]		

(5)

D0	D1	D2	D3	D4
---------α -------		β	—– –––– γ -------	
Dmin	D$n-1$	**D**n	D$n+1$	Dmax
		[ab]		
	[a]	[ac]	[abc]	
	[b]	[ad]	[abd]	
[ ]	[c]	[bc]	[acd]	[abcd]
	[d]	[bd]	[bcd]	
		[cd]		
Dmin	Dmin+1	Dn	Dmax −1	**D**max
μ	-------------λ ——————------			K

(6) at the bottom of (5):

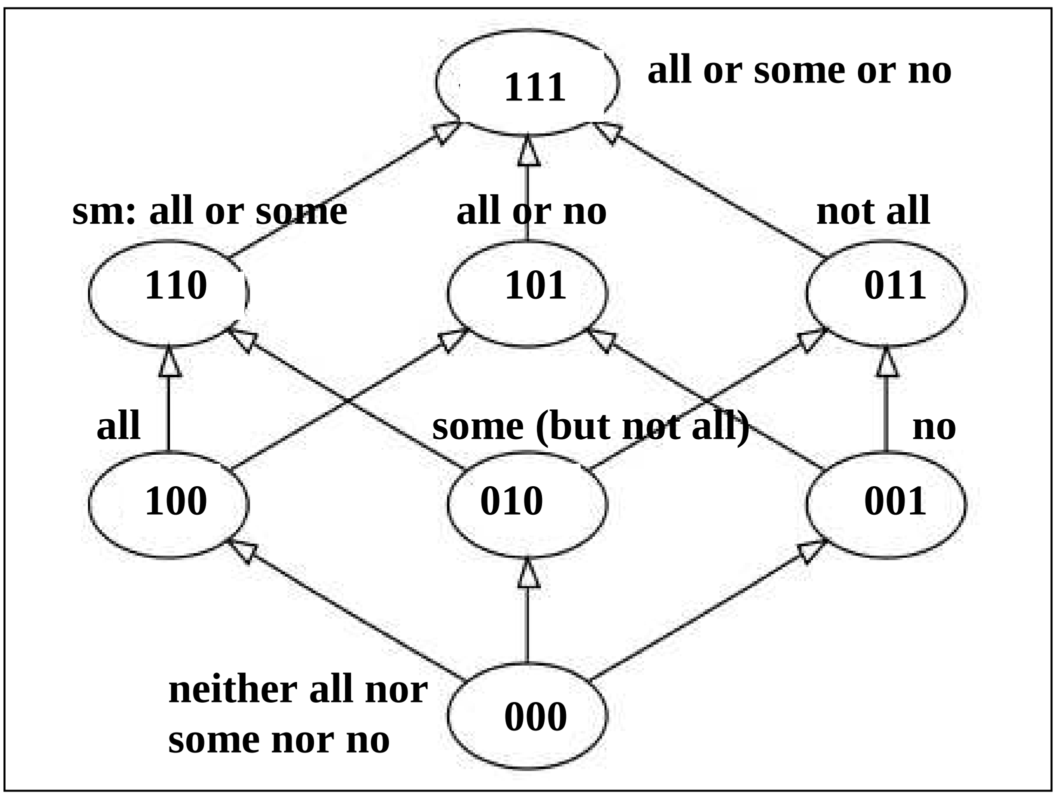
$\kappa \equiv Dmax = \{X \subseteq U : |X \cap D| = |D|\}$
$= \{X \subseteq U : D \subseteq X\}$
$= \{X \subseteq U : X \cap D = \emptyset\}$ (23a)

$\lambda \equiv Dmin{+}1 \cup \ldots \cup Dmax{-}1 = \{X \subseteq U : 0 < |X \cap D| < |D|\}$ (23b)

$\mu \equiv \quad Dmin \quad = \{X \subseteq U : |X \cap D| = 0\}$

"The bottom end μ denotes the set of all sets which do not intersect with the domain of quantification D, whereas the top end κ refers to the set of all sets which completely include the domain set D."

(7) Smessaert uses a shorthand notation format in the shape of a string of three bit positions. A value 1 for a particular area means that it is part of the quantifier denotation, whereas a value 0 indicates that it is not.



level 2

level 1

Hasse-diagram (Predicate Logic): 8 vertices

(8)

level 1 quantifiers (**L1**) $\kappa\lambda\mu$

A-corner[1]: [[all (books)]] $\equiv$ 100 = κ

Y-corner:[[some but not all (henceforth *some*) (books)]] $\equiv$ 010 = λ

E-corner: [[no (books)]] $\equiv$ 001 = μ

level 2 quantifiers (**L2**) $\kappa\lambda\mu$

I-corner: [[some (henceforth *sm*) (books)]] $\equiv$ 110 = $\kappa \cup \lambda$

U-corner: [[no or all (books)]] $\equiv$ 101 = $\kappa \cup \mu$

O-corner: [[not all (books)]] $\equiv$ 011 = $\lambda \cup \mu$

(9) Any L1 quantifier has a unique L2 quantifier as its contradictory and vice versa; any two L1 quantifiers are one anothers contraries; and any two L2 quantifiers are one anothers subcontraries. Here are the formal definitions in bitstring notation.

Contradictory(q1,q2) iff ([[q1]] $\cap$ [[q2]] = 000) *and* ([[q1]] $\cup$ [[q2]] = 111)

CD(no, sm) (001 $\cap$ 110 = 000) *and* (001 $\cup$ 110 = 111)

CD(all, not all) (100 $\cap$ 011 = 000) *and* (100 $\cup$ 011 = 111)

CD(some, no or all) (010 $\cap$ 101 = 000) *and* (010 $\cup$ 101 = 111)

Contrary(q1,q2) iff ([[q1]] $\cap$ [[q2]] = 000) *and* ([[q1]] $\cup$ [[q2]] $\neq$ 111)

CR(no, some) (001 $\cap$ 010 = 000) *and* (001 $\cup$ 010 $\neq$ 111)

CR(no, all) (001 $\cap$ 100 = 000) *and* (001 $\cup$ 100 $\neq$ 111)

CR(some, all) (010 $\cap$ 100 = 000) *and* (010 $\cup$ 100 $\neq$ 111)

Subcontrary(q1,q2) iff ([[q1]] $\cap$ [[q2]] $\neq$ 000) *and* ([[q1]] $\cup$ [[q2]] = 111)

SCR(not all, sm) (011 $\cap$ 110 $\neq$ 000) *and* (011 $\cup$ 110 = 111)

SCR(not all, all or no) (011 $\cap$ 101 $\neq$ 000) *and* (011 $\cup$ 101 = 111)

SCR(some, all or no) (110 $\cap$ 101 $\neq$ 000) *and* (110 $\cup$ 101 = 111)

Entail(q1, q2) iff ([[q1]] $\cap$ [[q2]] = [[q1]]) *and* ([[q1]] $\cup$ [[q2]] = [[q2]])

iff [[q1]] $\subseteq$ [[q2]] (19)

Entailment

ENT(all, sm) 100 $\subseteq$ 110

ENT(all, no or all) 100 $\subseteq$ 101

ENT(some, sm) 010 $\subseteq$ 110

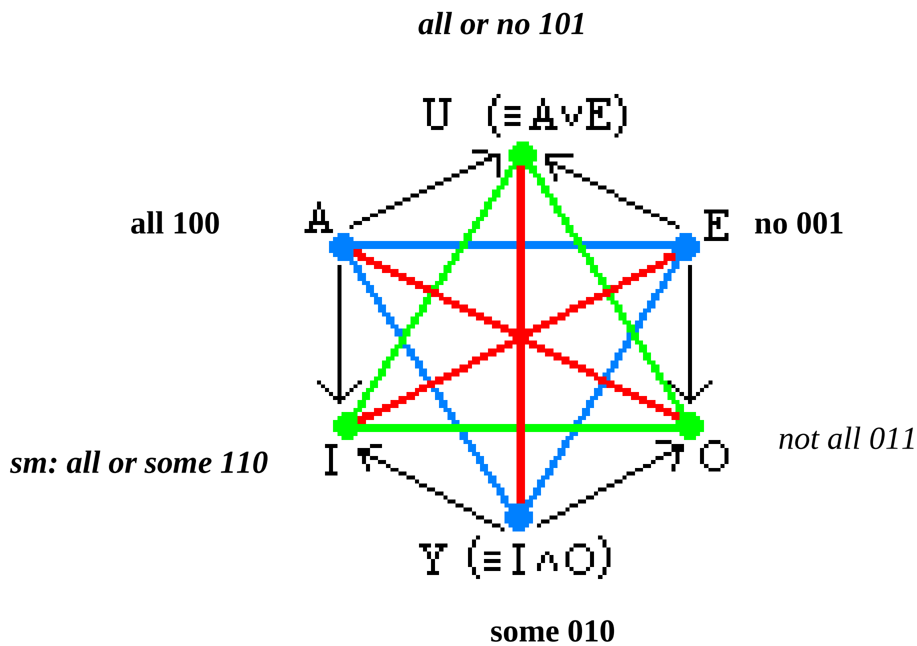
ENT(some, not all) 010 $\subseteq$ 011

ENT(no, not all) 001 $\subseteq$ 011

ENT(no, no or all) 001 $\subseteq$ 101

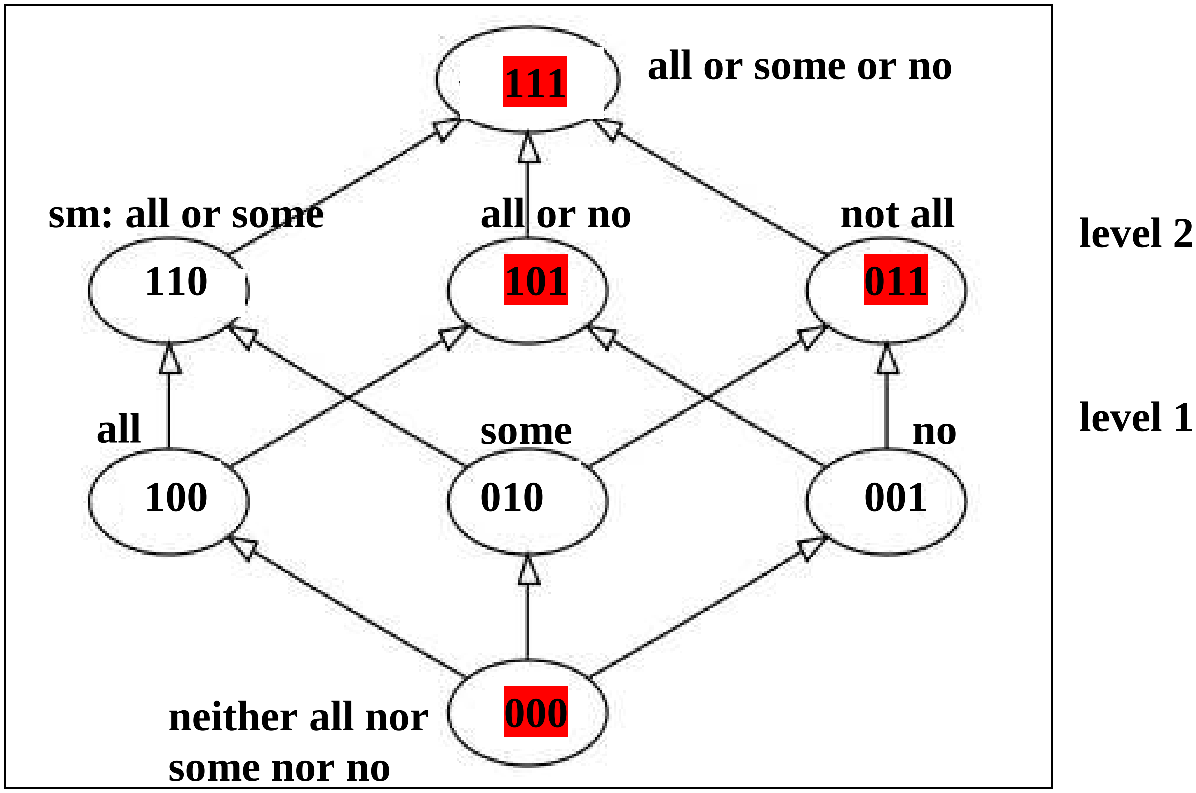
[1] The letters of the corners refer to their names in the Boethian Square of Oppositions and the Blanché (1969) star (cf. below).

(10)
This setup can be represented by means of a bitriangular representation, a so-called Blanché-star (Blanché 1969), where contradictories are connected by red lines, contraries by blue lines and subcontraries by green lines.



Blanché star (Predicate Logic): 6 vertices (111 and 000 missing)

(11)
Looking at matters from a linguistic viewpoint, it is to be noted that the enriched representation by means of a Hasse diagram has four corners for which lexicalization by means of a single term is nonexistent or extremely rare, namely 111, 000, 011 and 101:



The fact that 000 and 111 do not lexicalize as a single word is because the predicates involved are trivial, i.e. completely non-informative and can therefore not serve for contingent situations (it is logically necessary that "all or some or no flags are green"). The other corners that do not lexicalize (with maybe a chance exception if Seuren is right) are both secondary operators whose intersection is the E-corner operator, which itself is the least often lexicalized of the level 1 corners cross-linguistically (cf. Horn 1989). Somehow, negative corners are less easily lexicalized or only non-naturally so (as in the case of the scientifically constructed O-corner item *nand*.

2. The mereological algebra of colours

(12) The Boolean definitions of the Aristotelian relations of Opposition straightforwardly carry over to the primary and secondary colours, modulo replacement of

(a) settheoretical union by mereological sum (⊕), the individuals involved in the operation being wavelengths of visible light (or alternatively activation of the cone in the retina that is sensitive to that particular wavelength). Hence, the mereological sum is the combination of the wavelengths of visible light in question, which yields a different colour. For example, the mereological sum of RED and GREEN yields YELLOW, which is indeed a combination of the wavelengths of visible light of RED and GREEN. Note the difference between such a mereological sum and settheoretical union: a description of the settheoretical union of RED and GREEN would be "RED or GREEN or the combination of RED and GREEN (i.e. YELLOW)". A mereological sum does not include the first two disjuncts (RED, GREEN) but only the combination, the reason being that mereology is interested in nontrivial part-whole relationships, and YELLOW is the only nontrivial holonym for the meronyms RED and GREEN.

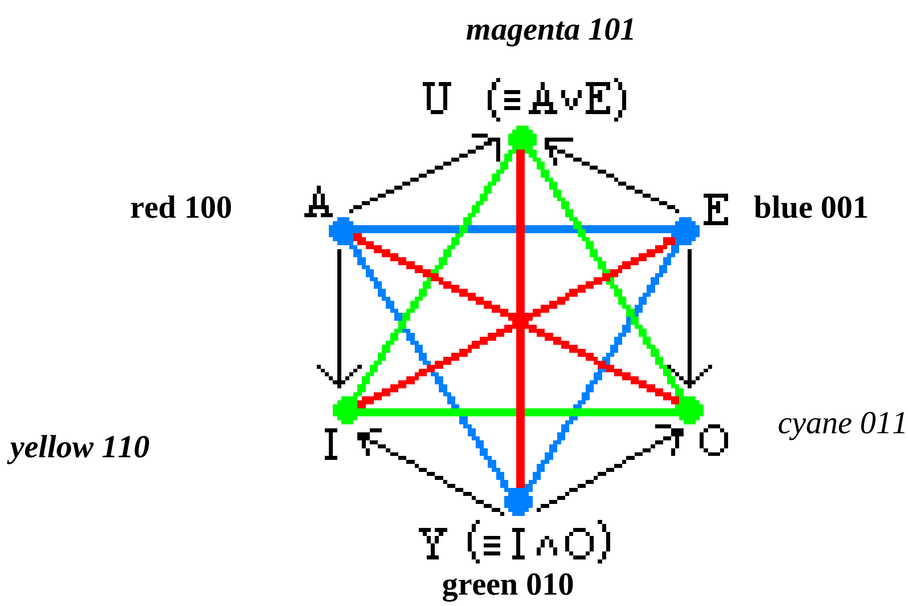
(b) settheoretical intersection by mereological product (⊗), which amounts to reducing the individuals involved to the wavelength(s) of visible light they have in common. This is what happens when we mix colours, which amounts to removing or blocking the wavelengths that the colours one mixes do not share. For example, when we mix YELLOW (which is the mereological sum of the wavelengths of RED and GREEN as we saw above) and MAGENTA (which is the mereological sum of RED and BLUE), we end up with what they share: RED;

(c) quantifiers by mereological individuals, i.c. colours such as RED, GREEN, etc.;

(d) The settheoretical null set and universe by BOTTOM and TOP respectively, which are individuals in their own right. In the colour algebra they are respectively BLACK and WHITE. Note that there is often controversy about the status of a BOTTOM in a mereological system. Thus, one might argue in our case that BLACK is qualitatively different from all the rest in that it is really the *absence* of cone activity and therefore only an "individual" if one reifies the absence of activation of any cone due to the absence of any wavelength of visible light into something. But we do see BLACK of course, so the idea that there is a BOTTOM is justified. The only problem that poses from the viewpoint of naturalness is that BLACK trivially a "part" of RED (and any other colour), parallel to the way in which the null set is a subset of any set.

So, let's have a look at the resulting algebra, which turns out to be perfectly isomorphic to the bitstring analysis for the quantifiers of predicate logic above:

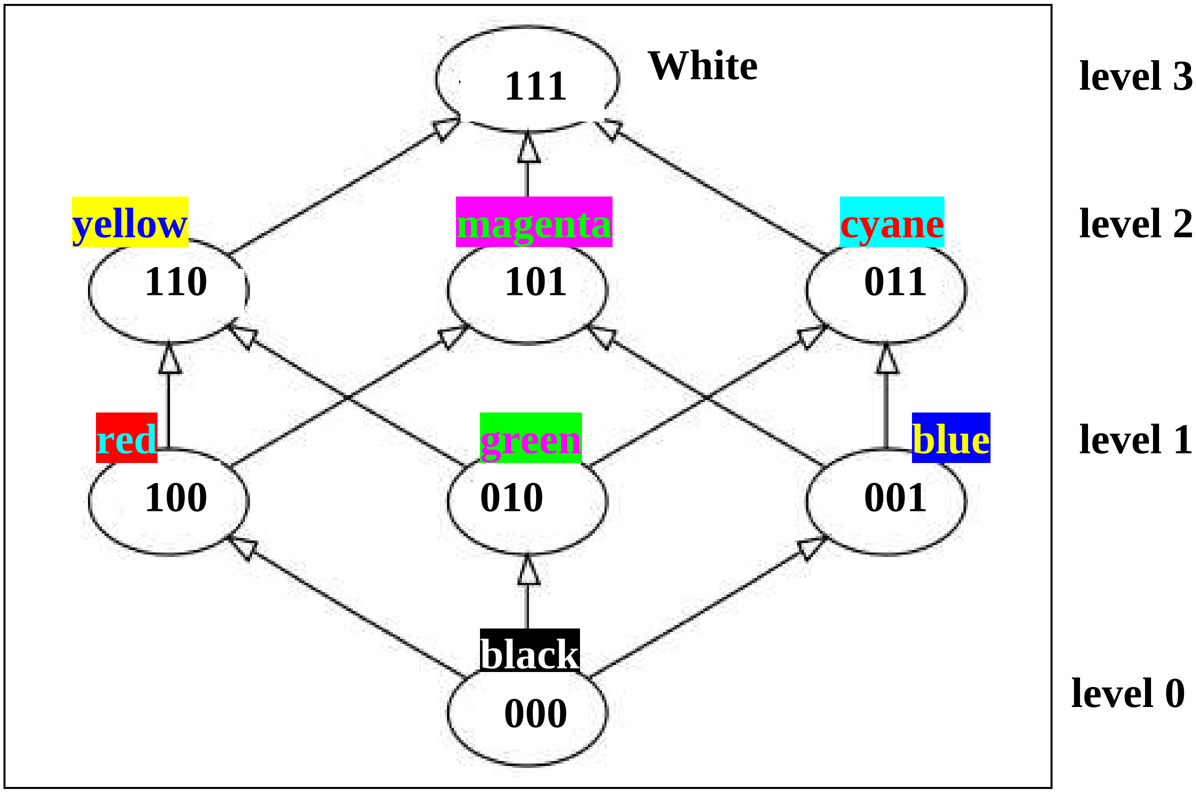
(13)
level 1 (primary) colours (L1) $\kappa\lambda\mu$
A-corner: RED $\equiv$ 100 = κ
Y-corner: GREEN $\equiv$ 010 = λ
E-corner: BLUE $\equiv$ 001 = μ
level 2 (secondary) colours (L2) $\kappa\lambda\mu$
I-corner: YELLOW $\equiv$ 110 = $\kappa \oplus \lambda$ (25b)
U-corner: MAGENTA $\equiv$ 101 = $\kappa \oplus \mu$ (25c)
O-corner: CYANE $\equiv$ 011 = $\lambda \oplus \mu$ (25a)



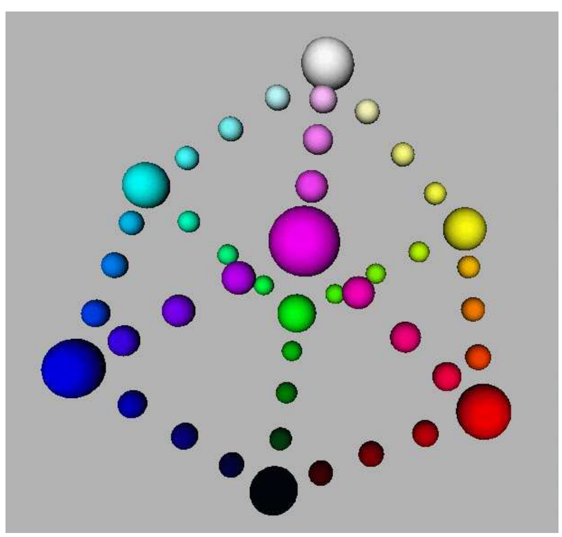
Blanché star (Colours): 6 vertices (white (111) and black (000) missing)

(14) two colours have to be added:
Level 0 (BOTTOM) colour (L0): BLACK $\equiv$ 000 = $\kappa \otimes \lambda \otimes \mu$
Level 3 (TOP) colour (L3): WHITE $\equiv$ 111 = $\kappa \oplus \lambda \oplus \mu$

(15) A Hasse-diagram can easily accommodate these two new vertices.



(16) A threedimensional version: the following colour cube developed by Hans Smessaert (actually, its mirror image: it has red and blue switched and hence also yellow and cyane, but for the cubic nature of the object that is immaterial):



Smessaert (5/1/2010) - The colour cube

(17)
Any L1 colour has a unique L2 colour as its negate or complementary (which is the mereological counterpart of *contradictory* in logic) and vice versa, any two L1 colours are one anothers "contraries" (= nonoverlapping colours whose mereological sum is not the mereological TOP colour WHITE (111)), and any two L2 colours are one anothers "subcontraries",i.e. mutually partially overlapping colours whose mereological sum is the mereological TOP colour WHITE.

Negate (or "complementary"): the negate of A, NEG(A), is that individual whose parts are exactly those that are discrete from A. The negate is the mereological counterpart of contradictoriness in logic. In the mereological algebra of colours the negate of a colour is what is known as its complementary colour.

Negate/Complementary colour: (q1,q2) iff ([[q1]] $\otimes$ [[q2]] = 000) *and* ([[q1]] $\oplus$ [[q2]] = 111)
CD(BLUE, YELLOW) (001 $\otimes$ 110 = 000) *and* (001 $\oplus$ 110 = 111)
CD(RED, CYANE) (100 $\otimes$ 011 = 000) *and* (100 $\oplus$ 011 = 111)
CD(GREEN, MAGENTA) (010 $\otimes$ 101 = 000) *and* (010 $\oplus$ 101 = 111)
CD(BLACK, WHITE) (000 $\otimes$ 111 = 000) *and* (000 $\oplus$ 111 = 111)

Mereological "contraries": (q1,q2) iff ([[q1]] $\otimes$ [[q2]] = 000) *and* ([[q1]] $\oplus$ [[q2]] $\neq$ 111)
CR(BLUE, GREEN) (001 $\otimes$ 010 = 000) *and* (001 $\oplus$ 010 $\neq$ 111)
CR(BLUE, RED) (001 $\otimes$ 100 = 000) *and* (001 $\oplus$ 100 $\neq$ 111)
CR(GREEN, RED) (010 $\otimes$ 100 = 000) *and* (010 $\oplus$ 100 $\neq$ 111)

Mereological "subcontraries" (q1,q2) iff ([[q1]] $\otimes$ [[q2]] $\neq$ 000) *and* ([[q1]] $\oplus$ [[q2]] = 111)
SCR(CYANE, YELLOW) (011 $\otimes$ 110 $\neq$ 000) *and* (011 $\oplus$ 110 = 111)
SCR(CYANE, MAGENTA) (011 $\otimes$ 101 $\neq$ 000) *and* (011 $\oplus$ 101 = 111)
SCR(YELLOW, MAGENTA) (110 $\otimes$ 101 $\neq$ 000) *and* (110 $\oplus$ 101 = 111)

Mereological "Entailment" = Proper parthood PP[2] (q1, q2) iff ([[q1]] $\otimes$ [[q2]] = [[q1]]) *and* ([[q1]] $\oplus$ [[q2]] = [[q2]])
iff [[q1]] $\subseteq$ [[q2]]

Proper Parthood
PP(RED, YELLOW) 100 $\subset$ 110
PP(RED, MAGENTA) 100 $\subset$ 101
PP(GREEN, YELLOW) 010 $\subset$ 110

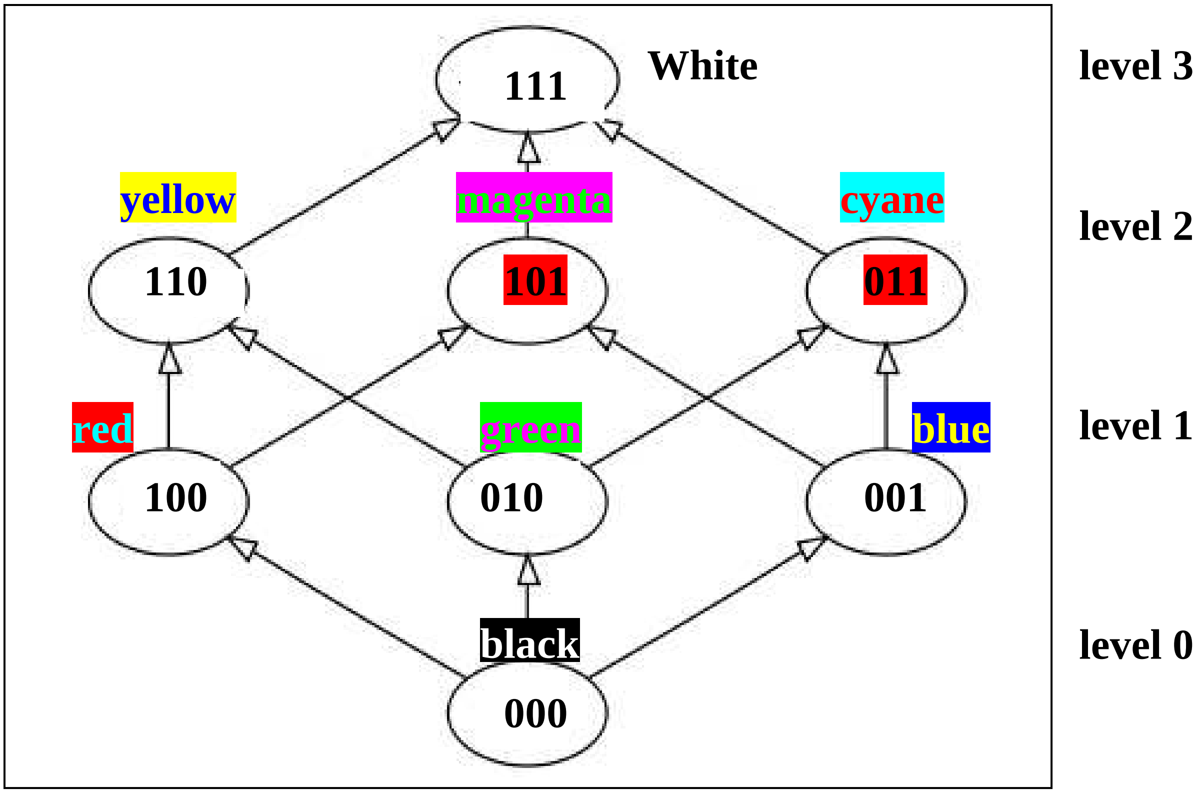
[2] Instead of using the equivalent of set inclusion, mereology employs the equivalent of PROPER set inclusion (otherwise one could say that RED is a part of itself, which stretches our natural intuition of part-whole (meronym – holonym) relations. In this respect, mereology is like natural set theory in Seuren's sense.

PP(GREEN, CYANE) 010 $\subset$ 011
PP(BLUE, CYANE) 001 $\subset$ 011
PP(BLUE, MAGENTA) 001 $\subset$ 101

PP(RED, WHITE) 100 $\subset$ 111
PP(GREEN, WHITE) 010 $\subset$ 111
PP(BLUE, WHITE) 001 $\subset$ 111

PP(YELLOW, WHITE) 110 $\subset$ 111
PP(MAGENTA, WHITE) 101 $\subset$ 111
PP(CYANE, WHITE) 011 $\subset$ 111

(18) Let us look at the linguistic side of the matter now and consider the status of the colour names in the different corners from the perspective of natural vs. non-natural (or alternatively nonexisting) lexicalization. In the colour algebra, the enriched representation by means of a Hasse diagram has only two corners for which lexicalization by means of a single term is not a basic natural colour term, namely 011 (cyane) and 101 (magenta), exactly the equivalents of the two level two corners which resisted lexicalization in the algebra for predicate logical operators. The other two that were not lexicalisable in the predicate calculus, are now the locuses of white (111) and black (000), respectively. This is a direct consequence of the difference between a mereological sum and set-theoretical union: whereas 111 denotes the whole universe in the case of predicate logic, it does not denote the whole universe of colours in a mereology (which would amount to "RED or GREEN or BLUE, etc.). Just as the mereological sum of GREEN and RED only denotes the combination of the wavelengths of GREEN and RED and not the two primaries that enter into it, white only denotes the mereological sum of chromatic colours, but none of those chromatic colours themselves. In that sense the mereological sum contains a combination of chromatic colours, but does not denote the colours that enter into it. The enriched diagram with two extra vertices beyond those available in the Blanché star is therefore crucial for the representation of the colour algebra. Not so for predicate logic, for reasons of noninformativity specified earlier.



Hasse-diagram (Colours): 8 vertices

3. Conclusion

The algebras of predicate logic and colours are perfectly isomorphic. The natural predicate logic of language has a cognitively deeper counterpart, the natural logic of colours.

4. Natural language application

4.1 natural versus non-natural quantifiers (*nand/*nall) and colour terms (yellow vs. "cyane")

- Evolution sequence of natural terms for quantifier and colour oppositions

  The incremental sequence for predicate logic operators as worked out in Jaspers (2005) along Peircean lines on the basis of the operator NEC is very similar to the Berlin-Kay (1969) incremental sequence for cross-linguistic colour term systems. They observed that if a language has three colour terms, they will always be dark/black, light/white and red; if another term comes in, it will be green or yellow, languages with one more colour will have the one they did not have yet (yellow or green) as the next one, and only languages who have yet another colour term will have blue, a term which many languages do not possess (cp. the relative infrequency of E-corner items noted by Horn 1989)

	1	2	3	4
NEC	> sm (or all)	> all	> some (but not all)	> no
“dark” (BLACK+BLUE)	“light” (YELLOW+WHITE+…)	“red” (RED)	“green” (GREEN)	“blue” (BLUE)
			“yellow” (YELLOW)	

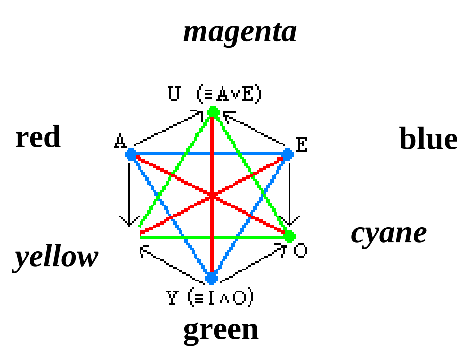
4.2 cognitive complexity and the generalized O-corner problem

There are no natural names for colours 101(magenta) and 011 (cyane), just as there are no natural simplex names for quantifiers 101 (all or no) and 011 (*nand/*nall). General observation: the basic colours that attempt to incorporate 001 (BLUE/NO) have lexicalization trouble or are cognitively less accessible.

APPENDIX 1: Colours, Squares and Triangles – some theoretical background notions.

In the debate between the defenders of the classical perspective on the logical square of oppositions on the one hand and proponents of a triangular approach on the other (Hamilton 1860, Ginzberg 1913, Jespersen 1924: 324-325, Horn 1989: 253, Seuren 2002: 20-21), an original compromise was reached in work by Jacoby (1950), Blanché (1953, 1969), Hegenberg (1957) and Sesmat (1951). Their proposals took the form of a logical hexagon in Blanché's work, in fact a bitriangle derived by supplementing the AYE-triangle of contraries with its dual IOU, a triangle of subcontrary relations (Moretti 2009, 142; Horn 1989; Béziau 2003). The latter system was applied to several arguably bitriangular conceptual fields.

The hypothesis elaborated above is that there is yet another bitriangle which can be fruitfully mapped onto the hexagon, namely that of the six colours *red, green, blue, yellow, magenta* and *cyane* and more specifically of the percepts they embody. In view of Thomas Young's trichromatic theory of colour vision and its later refinements, it was shown that the triangle of contraries consists of the primary colours *red-green-blue*, with *red* (= L: Longest wavelength) in the A-corner, M(id wave) *green* in the Y-corner and S(hort wave) *blue* in the E-corner. The remaining three colours are secondary in that at the physical and perceptual level they represent the combined wavelengths – in mereological terms: the sum (comparable but not identical to set-theoretical union) of the wavelengths – of red and green (= yellow), green and blue (= cyane) and blue and red (= magenta) respectively.



As is well-known from the phenomenon of afterimages (and aspects of Hering's opponent process theory), the secondary triad *yellow-cyane-magenta* consists of complementaries of the primary colours, a pattern of opposition that was shown above to be the colour equivalent of contradictoriness relations in the logical square (and hexagon) of oppositions.

This is not to say that there are no differences between the logical and the chromatic stars. For one thing, without modification the star is still blatantly deficient for the colour system in that such basic colours as *black* and *white* have no position in it, while it is less immediately obvious that additional vertices are required for the logical incarnation of the hexagon. And how can sense be made of the entailment arrows in the colour embodiment of the star, given that at least for colour *terms* it is fairly natural to maintain that they are simply all contrary predicates and that neither contradictoriness nor entailment have a role to play in their lexical field?

To solve the first issue, an appeal is made to a 3D cube or its 2D counterpart, a Hasse Diagram, as an alternative to Blanché's 2D- star-like model. This will enable the introduction of the achromatic colours *white* and *black* into the system, occupying the two extra vertices, more specifically the lattice-theoretical top and bottom of the expanded geometrical figure.

Secondly, the entailment problem was addressed by elaborating a point that remained largely implicit in the above account, namely that the relevant system of colour oppositions does not concern the synchronic system of colour *terms*, which from a logical perspective is a lexical field of contrary predicates. Rather, the oppositions play out at the level of the physics of colour and the nature of colour perception and are consequently mereological rather than set-theoretical. This insight can not only be illustrated in the additive RGB-system of colour emission but also by switching to the subtractive colour system which is operative when we mix paint, for instance. This switch has the same properties as conversion in logic. Thus, colour mixing amounts to the suppression (mereologically, the product instead of the sum) of certain wavelengths. Analogously to the way conversion works in standard logical systems, conversion from the additive to the subtractive colour system means that the secondary triad, that of the "subcontrary" colours *yellow, magenta, cyane* now becomes the primary one and that the operation employed to compose the remaining three corners by mixing also switches, in casu from wavelength sum to wavelength product, where sum and product are the known mereological counterparts of union and intersection in set-theory and of logical disjunction and logical conjunction in the propositional calculus. To summarize, there is an asymmetric mereological containment relationship (cp. proper inclusion in set-theory) between colours such as *red* and *yellow*, an asymmetrical relationship which is at the root of the sense of reversal when we switch from the additive to the subtractive system and which is the mereological equivalent of entailment in standard logic.

On the whole, the patterning of different kinds of colour oppositions (complementary vs. noncomplementary colours, chromatic versus non-chromatic colours, contrary (meronymical) colours versus subcontrary (holonymical) colours) and of semantic oppositions in logic is isomorphic on so many counts that it would be extremely surprising if there was no cognitive algorithm common to and serving both domains. An important question that arises in view of this isomorphy of physical/physiological patterns of colour opposition and natural logical patterns of opposition more traditionally mapped onto triangles, squares and stars is the issue of whether the opposition pattern represents a separate cognitive module that feeds these two different cognitive domains or whether it originated in one or the other faculty first and was later utilized by the other. Whatever the answer to this issue, the present findings argue against extreme Whorfian relativism. The oppositions noted and defined are perfectly equivalent to those in logic, for which it would be hard to maintain that they are not universal, given that nobody has ever come up with a language that does not have quantifiers or propositional operators. And as far as cognitive complexity and linguistic ease of lexicalization is concerned: could it be an accident that it is the U and the O-corners of the star which are never lexicalized (possibly with a chance exception for the O-corner as suggested by Pieter Seuren) in natural logic and get correspondingly nonnatural, i.e. constructed or scientific, names such as *cyane* and *magenta* in the colour cube? Don't think so (and will later provide arguments based on experiments with Munsell colour chips).

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